

Do gravitational waves carry energy-momentum? A reappraisal

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After direct detection gravitational radiation in 2015 many authors are publishing remakes of their old articles about this radiation. I decided to follow this line in my Lecture delivered at the Conference “Varcosmofun’16” (12-17 September 2016, Szczecin, Poland, EU). Namely, I have presented at this Conference an updated summary of my past articles on gravitational radiation. As a base for my presentation I have used mainly the article published in 2002 in *Annalen der Physik* [1] and the articles [2].

In these past articles I have showed that the real gravitational waves which possess a non-vanishing Riemann tensor always carry energy-momentum (and also angular momentum). Our proof have used canonical superenergy and supermomentum tensor for gravitational field in former articles and the averaged relative energy-momentum tensor in latter. In this article we confine to the energy-momentum only.

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I. INTRODUCTION

In General Relativity (**GR**) the gravitational field Γ^i_{kl} does not possess any energy-momentum tensor. Instead, it only possesses the so-called “energy-momentum pseudotensors”. In fact, this is a consequence of the Einstein Equivalence Principle (**EPP**). Because of that, many authors put in doubt the reality of the energy-momentum (and the angular momentum also) transfer by gravitational waves. As the main argument, some of these authors used the fact that for the majority of exact solutions of the vacuum Einstein field equations which represent gravitational waves, energy-momentum pseudotensors *globally vanish* in certain coordinates. In consequence, these pseudotensors give “no gravitational energy and no gravitational energy flux” in these coordinates. Some other authors argue that the vanishing of the components ${}_g t^{ok}$ (or ${}_g t_o^k$) of the gravitational pseudotensor ${}_g t^{ik}$ (or ${}_g t_i^k$) may be treated as a coordinate condition coupled to the Einstein equations and yield (in special coordinates) “global vanishing of the pure gravitational energy and the pure gravitational energy flux”.

However, *such conclusions are physically incorrect* because they rely on non-tensorial, *coordinate dependent expressions* (See [1] for full argumentation).

The energy and the energy flux (as well as the angular momentum) of the real gravitational field which has $R_{iklm} \neq 0$ *always exist and do not vanish*. In order to show this, one should use the coordinate independent expressions like our *canonical superenergy tensor* for gravitational field (used in our older articles) or *canonical averaged relative energy-momentum tensor* for gravitational field (used in our latter articles).

II. THE CANONICAL GRAVITATIONAL SUPERENERGY TENSOR AND THE CANONICAL AVERAGED RELATIVE GRAVITATIONAL ENERGY-MOMENTUM TENSOR

As it was already mentioned, in the framework of general relativity (**GR**) the gravitational field *has non-tensorial strengths* $\Gamma^i_{kl} = \{^i_{kl}\}$ and *admits no energy-momentum tensor*. One can only attribute to this field *gravitational energy-momentum pseudotensors*. The leading object of such a kind is the *canonical gravitational energy-momentum pseudotensor* ${}_E t_i^k$ proposed already in past by Einstein. This pseudotensor is a part of the *canonical energy-momentum complex* ${}_E K_i^k$ in **GR**.

The canonical complex ${}_E K_i^k$ can be easily obtained by rewriting Einstein equations to the superpotential form

$${}_E K_i^k := \sqrt{|g|} (T_i^k + {}_E t_i^k) = {}_F U_i^{[kl]}{}_{,l} \quad (1)$$

where $T^{ik} = T^{ki}$ is the symmetric energy-momentum tensor for matter, $g = \det[g_{ik}]$, and

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$$\begin{aligned}
{}_E t_i^k &= \frac{c^4}{16\pi G} \{ \delta_i^k g^{ms} (\Gamma_{mr}^l \Gamma_{sl}^r - \Gamma_{ms}^r \Gamma_{rl}^l) \\
&\quad + g_{,i}^{ms} [\Gamma_{ms}^k - \frac{1}{2} (\Gamma_{tp}^k g^{tp} - \Gamma_{tl}^l g^{kt}) g_{ms} \\
&\quad - \frac{1}{2} (\delta_s^k \Gamma_{ml}^l + \delta_m^k \Gamma_{sl}^l)] \}; \\
{}_F U_i^{[kl]} &= \frac{c^4}{16\pi G} g_{ia} (\sqrt{|g|})^{(-1)} [(-g) (g^{ka} g^{lb} - g^{la} g^{kb})]_{,b}.
\end{aligned} \tag{2}$$

${}_E t_i^k$ are components of the canonical energy-momentum pseudotensor for gravitational field $\Gamma_{kl}^i = \{\frac{i}{kl}\}$, and ${}_F U_i^{[kl]}$ are von Freud superpotentials.

$${}_E K_i^k = \sqrt{|g|} (T_i^k + {}_E t_i^k) \tag{3}$$

are components of the *Einstein canonical energy-momentu complex*, for matter and gravity, in **GR**.

In consequence of (1) the complex ${}_E K_i^k$ satisfies local conservation laws

$${}_E K_i^k{}_{,k} \equiv 0. \tag{4}$$

In very special cases and in special coordinates, one can obtain from these local conservation laws the reasonable integral conservation laws of the energy and momentum.

Despite that one can easily introduce in **GR** the *canonical superenergy tensor* for gravitational field. This was done in past in a series of our articles (See, e.g., [3] and references therein). It appeared that the idea of the superenergy tensors is universal: to any physical field having an energy-momentum tensor or pseudotensor one can attribute the corresponding superenergy tensor.

So, let us give a short reminder of the general, constructive definition of the superenergy tensor S_a^b applicable to gravitational field and to any matter field. The definition uses *locally Minkowskian structure* of the spacetime in **GR** and, therefore, it fails in a spacetime with torsion, e.g., in Riemann-Cartan spacetime. In the normal Riemann coordinates **NRC(P)** we define (pointwise)

$$S_{(a)}^{(b)}(P) = S_a^b := (-) \lim_{\Omega \rightarrow P} \frac{\int_{\Omega} [T_{(a)}^{(b)}(y) - T_{(a)}^{(b)}(P)] d\Omega}{1/2 \int_{\Omega} \sigma(P; y) d\Omega}, \tag{5}$$

where

$$\begin{aligned}
T_{(a)}^{(b)}(y) &:= T_i^k(y) e_{(a)}^i(y) e_k^{(b)}(y), \\
T_{(a)}^{(b)}(P) &:= T_i^k(P) e_{(a)}^i(P) e_k^{(b)}(P) = T_a^b(P)
\end{aligned}$$

are *physical or tetrad components* of the pseudotensor or tensor field which describes an energy-momentum distribution, and $\{y^i\}$ are normal coordinates. $e_{(a)}^i(y), e_k^{(b)}(y)$ mean an orthonormal tetrad $e_{(a)}^i(P) = \delta_a^i$ and its dual $e_k^{(a)}(P) = \delta_k^a$ parallelly propagated along geodesics through P (P is the origin of the **NRC(P)**). We have

$$e_{(a)}^i(y) e_i^{(b)}(y) = \delta_a^b. \tag{6}$$

For a sufficiently small 4-dimensional domain Ω which surrounds **P** we require

$$\int_{\Omega} y^i d\Omega = 0, \quad \int_{\Omega} y^i y^k d\Omega = \delta^{ik} M, \tag{7}$$

where

$$M = \int_{\Omega} (y^0)^2 d\Omega = \int_{\Omega} (y^1)^2 d\Omega = \int_{\Omega} (y^2)^2 d\Omega = \int_{\Omega} (y^3)^2 d\Omega, \tag{8}$$

is a common value of the moments of inertia of the domain Ω with respect to the subspaces $y^i = 0$, ($i = 0, 1, 2, 3$). We can take as Ω , e.g., a sufficiently small analytic ball centered at P :

$$(y^0)^2 + (y^1)^2 + (y^2)^2 + (y^3)^2 \leq R^2, \quad (9)$$

which for an auxiliary positive-definite metric

$$h^{ik} := 2v^i v^k - g^{ik}, \quad (10)$$

can be written in the form

$$h_{ik} y^i y^k \leq R^2. \quad (11)$$

A fiducial observer \mathbf{O} is at rest at the beginning \mathbf{P} of the used Riemann normal coordinates $\mathbf{NRC}(\mathbf{P})$ and its four-velocity is $v^i = * \delta_o^i$. $= *$ means that an equations is valid only in special coordinates. $\sigma(P; y)$ denotes the two-point *world function* introduced in past by J.L. Synge [4]

$$\sigma(P; y) = * \frac{1}{2} (y^{o^2} - y^{1^2} - y^{2^2} - y^{3^2}). \quad (12)$$

The world function $\sigma(P; y)$ can be defined covariantly by the *eikonal-like equation* [4]

$$g^{ik} \sigma_{,i} \sigma_{,k} = 2\sigma, \quad \sigma_{,i} := \partial_i \sigma, \quad (13)$$

together with

$$\sigma(P; P) = 0, \quad \partial_i \sigma(P; P) = 0. \quad (14)$$

The ball Ω can also be given by the inequality

$$h^{ik} \sigma_{,i} \sigma_{,k} \leq R^2. \quad (15)$$

Tetrad components and normal components are equal at \mathbf{P} , so, we will write the components of any quantity attached to \mathbf{P} without tetrad brackets, e.g., we will write $S_a{}^b(P)$ instead of $S_{(a)}{}^{(b)}(P)$ and so on.

If $T_i{}^k(y)$ are the components of an energy-momentum tensor of matter, then we get from (5)

$${}_m S_a{}^b(P; v^l) = (2\hat{v}^l \hat{v}^m - \hat{g}^{lm}) \nabla_l \nabla_m \hat{T}_a{}^b = \hat{h}^{lm} \nabla_l \nabla_m \hat{T}_a{}^b. \quad (16)$$

Hat over a quantity denotes its value at \mathbf{P} , and ∇ means covariant derivative. Tensor ${}_m S_a{}^b(P; v^l)$ is *the canonical superenergy tensor for matter*.

For the gravitational field, substitution of the canonical Einstein energy-momentum pseudotensor as $T_i{}^k$ in (5) gives

$${}_g S_a{}^b(P; v^l) = \hat{h}^{lm} \hat{W}_a{}^b{}_{lm}, \quad (17)$$

where

$$\begin{aligned} W_a{}^b{}_{lm} = & \frac{2\alpha}{9} [B_{alm}^b + P_{alm}^b \\ & - \frac{1}{2} \delta_a^b R^{ijk}{}_m (R_{ijkl} + R_{ikjl}) + 2\delta_a^b \beta^2 E_{(l|g} E_{|m)}^g \\ & - 3\beta^2 E_{a(l|} E_{|m)}^b + 2\beta R_{(a|g|l)}^b E_{m)}^g]. \end{aligned}$$

Here $\alpha = \frac{c^4}{16\pi G} = \frac{1}{2\beta}$, and

$$E_i{}^k := T_i{}^k - \frac{1}{2} \delta_i^k T \quad (18)$$

is the modified energy-momentum tensor of matter [10]. On the other hand

$$B_{alm}^b := 2R^{bik}{}_{(l|} R_{aik|m)} - \frac{1}{2} \delta_a^b R^{ijk}{}_l R_{ijkm} \quad (19)$$

are the components of the *Bel-Robinson tensor* (**BRT**), while

$$P_{alm}^b := 2R^{bik}_{(l|R_{aki|m})} - \frac{1}{2}\delta_a^b R^{jik}_{(l} R_{jkim)} \quad (20)$$

is the Bel-Robinson tensor with “transposed” indices (ik) . Tensor ${}_g S_a^b(P; v^l)$ is the *canonical superenergy tensor* for gravitational field $\{^i_{kl}\}$. In vacuum ${}_g S_a^b(P; v^l)$ takes the simpler form

$${}_g S_a^b(P; v^l) = \frac{8\alpha}{9} \hat{h}^{lm} (\hat{C}^{bik}_{(l} \hat{C}_{aik|m})} - \frac{1}{2} \delta_a^b \hat{C}^{i(kp)}_{(l} \hat{C}_{ikp|m})}. \quad (21)$$

Here C^a_{blm} denote components of the *Weyl tensor*.

Some remarks are in order:

1. In vacuum the quadratic form ${}_g S_a^b v^a v_b$, where $v^a v_a = 1$, is *positive-definite* giving the gravitational *superenergy density* ϵ_g for a fiducial observer **O** which is at rest at the beginning **P** of the **NRC(P)**.
2. In general, the canonical superenergy tensors are uniquely determined only along the world line of the observer **O**. But in special cases, e.g., in Schwarzschild spacetime or in Friedman universes, when there exists a physically and geometrically distinguished four-velocity $v^i(x)$, one can introduce in an unique way the unambiguous fields ${}_g S_i^k(x; v^l)$ and ${}_m S_i^k(x; v^l)$.
3. We have proposed in our previous papers to use the tensor ${}_g S_i^k(P; v^l)$ as a substitute of the non-existing gravitational energy-momentum tensor.
4. It can easily seen that the superenergy densities $\epsilon_g := {}_g S_i^k v^i v_k$, $\epsilon_m := {}_m S_i^k v^i v_k$ for an observer **O** who has the four-velocity v^i correspond exactly to the *energy of acceleration* $\frac{1}{2} m \vec{a} \vec{a}$ which is fundamental in Appel’s approach to classical mechanics [5].

In past we have used the canonical superenergy tensors ${}_g S_i^k$ and ${}_m S_i^k$ to local (and also, in some cases, to global) analysis of well-known solutions to the Einstein equations like Schwarzschild and Kerr solutions; Friedman and Goedel universes, and Kasner and Bianchi I, II universes. The obtained results were interesting (See [3]).

We have also studied the transformational rules for the canonical superenergy tensors under conformal rescaling of the metric $g_{ik}(x)$ [3, 6].

The idea of the superenergy tensors can be extended on angular momentum also [3]. The obtained angular superenergy tensors do not depend on a radius vector and they depend only on *spinorial part* of the suitable gravitational angular momentum pseudotensor. [11]

Changing the constructive definition (5) to the form

$$\langle T_a^b(P) \rangle := \lim_{\varepsilon \rightarrow 0} \frac{\int_{\Omega} \left[T_{(a)}^{(b)}(y) - T_{(a)}^{(b)}(P) \right] d\Omega}{\varepsilon^2 / 2 \int_{\Omega} d\Omega}, \quad (22)$$

where $\varepsilon := \frac{R}{L} > 0$ (equivalently $R = \varepsilon L$) is a real parameter and L is a dimensional constant : $[L] = m$, one obtains the *averaged relative energy-momentum tensors*. Namely, for matter one obtains

$$\langle {}_m T_a^b(P; v^l) \rangle = {}_m S_a^b(P; v^l) \frac{L^2}{6}, \quad (23)$$

and for gravity one obtains

$$\langle {}_g t_a^b(P; v^l) \rangle = {}_g S_a^b(P; v^l) \frac{L^2}{6}. \quad (24)$$

The components of the averaged relative energy-momentum tensors have correct dimensions but they depend on a dimensional parameter L which plays role of a fundamental length.

Of course, the fundamental length L must be infinitesimally small because its existence violates local Lorentz invariance. In [2] we have proposed a universal choose of the parameter L . Namely, we have proposed $L = 100L_P \approx 10^{-33}m$. Here $L_P := \sqrt{\frac{\hbar G}{c^3}} \approx 10^{-35}m$ is the *Planck length*. Following specialists in loop quantum gravity (**LQG**) our $L = 100L_P$ is approximately the smallest length over which the classical model of the spacetime is admissible.

As we can seen, the averaged energy-momentum tensors differ from the canonical superenergy tensors only by the constant multiplicator $\frac{L^2}{6}$, where L means some fundamental length. Thus, from the mathematical point of view these two kinds of tensors are equivalent. Physically they are not because their components have different dimensionality. Moreover, the averaged energy-momentum tensors depend on a fundamental length L . Owing to the last fact it seems that the canonical superenergy tensors are *more fundamental* than the canonical averaged relative energy-momentum tensors. This is the main reason why we have used (and still use) superenergy tensors in our papers. But one should emphasize that the canonical averaged relative energy-momentum tensors have an important superiority over the canonical superenergy tensors: their components have proper dimensions of the energy-momentum densities.

III. ENERGY AND MOMENTUM CARRYING BY GRAVITATIONAL WAVES

In order to prove that any real gravitational wave transfers energy-momentum we have used in our former papers the canonical superenergy tensor ${}_g S_i^k(P; v^l)$ for gravitational field.

By a direct calculation one can easily check that this tensor gives *positive-definite* superenergy density $\epsilon_s := {}_g S_i^k v^i v_k$ and a *non-vanishing* superenergy flux $P^i := (\delta_k^i - v^i v_k) {}_g S_l^k(P; v^a) v^l$ for every known solution of the vacuum Einstein equations which represents a real gravitational wave, i.e., a wave with $R_{iklm} \neq 0$. Here v^i means the four-velocity of an observer which is studying gravitational field and who is at rest in a **NRC(P)**. As examples we have considered in [1, 2] the following gravitational waves

1. Linearly polarized, plane gravitational wave in the coordinates (U, V, X, Y) in which the line element reads

$$ds^2 = 2(Y^2 - X^2) \frac{F(U)}{2} dU^2 + 2dUdV - dX^2 - dY^2, \quad (25)$$

where $F = F(U)$ is an arbitrary function;

2. Plane-fronted gravitational wave with parallel rays (p-p wave) having the following line element in the coordinates (U, V, X, Y)

$$ds^2 = 2H(X, Y, U) dU^2 + 2dUdV - dX^2 - dY^2, \quad (26)$$

where

$$\Delta H := \left(\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2} \right) H = 0. \quad (27)$$

The vector tangent to the V-lines is null and covariantly constant.

The p-p wave is a generalization of the plane wave.

3. The Einstein-Rosen (cylindrical) gravitational wave which has the following line element in cylindrical coordinates $x^0 = ct$, $x^1 = \varrho$, $x^2 = \varphi$, $x^3 = z$

$$ds^2 = e^{2(\gamma - \Psi)} (c^2 dt^2 - d\varrho^2) - \varrho^2 e^{-2\Psi} d\varphi^2 - e^{2\Psi} dz^2. \quad (28)$$

The metric functions $\gamma(x^0, x^1)$, $\Psi(x^0, x^1)$ satisfy the following system of partial differential equations

$$\begin{aligned} \Psi_{,11} + \frac{1}{\varrho} \Psi_{,1} - \Psi_{,00} &= 0, \\ \gamma_{,1} &= \frac{\varrho}{2} [(\Psi_{,1})^2 + (\Psi_{,0})^2], \\ \gamma_{,0} &= 2\varrho \Psi_{,0} \Psi_{,1}. \end{aligned}$$

For the all above gravitational waves we have obtained the positive definite superenergy densities and non-null superenergy fluxes (See [1, 2] for details).

The analogical result one gets also for any other real gravitational wave in full agreement with the remark 1 of the previous Section.

It results from this that the every gravitational wave, which has $R_{iklm} \neq 0$ *must also carry* the gravitational energy-momentum. If not, then there would be a contradiction between an “energy-momentum level” and a “superenergy level”, because our canonical, gravitational superenergy tensor originated as a kind of averaging in **NRC(P)** of the canonical gravitational energy-momentum pseudotensor.

The following quasilocal constructions confirm the above statements. Let us consider an observer \mathbf{O} which is studying gravitational field. His world-line is $x^a = x^a(s)$ and $\vec{v}: v^a = \frac{dx^a}{ds}$ represents his four-velocity. At any point \mathbf{P} of the world line one can define an instantaneous, local 3-space of the observer \mathbf{O} orthogonal to \vec{v} . This instantaneous 3-space has the following interior proper Riemannian metric

$$\gamma_{ab} := v_a v_b - g_{ab} = \star \left(\frac{g_{0a} g_{0b}}{g_{00}} - g_{ab} \right) = \star \gamma_{\alpha\beta} = \star(-) g_{\alpha\beta}, \quad (29)$$

where the Greek indices run over the values 1, 2, 3 (see e.g., [7]).

Then, by using the gravitational superenergy density ϵ_s and its flux P^i , one can easily construct in such instantaneous local 3-space the following expressions which have proper dimensions of the energy density and its flux

$$\epsilon_{en} := \oint_{S_2} \epsilon_s(P) d^2 S \approx \epsilon_s(P) \oint_{S_2} d^2 S = 4\pi R^2 \epsilon_s(P) > 0, \quad (30)$$

$$P^i := \oint_{S_2} P^i(P) d^2 S \approx P^i(P) \oint_{S_2} d^2 S = 4\pi R^2 P^i(P) \neq 0. \quad (31)$$

Here S_2 means an infinitesimal sphere $\gamma_{\alpha\beta} x^\alpha x^\beta = R^2$ in the instantaneous local 3-space of the observer \mathbf{O} centered on this observer.

The expressions (30)-(31) give us the *relative gravitational energy density* and its flux for an observer \mathbf{O} in his instantaneous 3-space orthogonal to \vec{v} .

In our latter articles we have used with the same goal as above the averaged relative gravitational energy-momentum tensor $<_g t_a^b(P; v^l) >$.

Namely, we defined

$$\epsilon_{en} := <_g t_a^b(P; v^l) v^a v_b, \quad (32)$$

$$P^i := (\delta_k^i - v^i v_k) <_g t_l^k(P; v^t) > v^l. \quad (33)$$

Here v^i mean, as usual, the 4-velocity components of an observer \mathbf{O} which is studying gravitational field.

From the fundamental properties of the gravitational superenergy and from the formulae (23)-(24) it is easily seen that

$$\epsilon_{en} > 0, \quad P^i \neq 0 \quad (34)$$

for every real gravitational wave.

Now, it seems to us that the using of the canonical relative gravitational energy-momentum tensor $<_g t_a^b(P; v^l) >$ in our considerations is more convincing than the using of the canonical gravitational superenergy tensor ${}_g S_a^b(P; v^l)$.

IV. CONCLUSION

If one wants to get correct information about energy-momentum (and angular momentum) of the real gravitational field by application of coordinate-dependent pseudotensors and complexes, then one has to use these strange objects in very special situations and coordinates. For example, one can use these objects to global analysis of a closed system in asymptotically flat, *Bondi-Sachs coordinates* [9]. In general one must use these objects locally in Riemann normal coordinates **NRC(P)** and extract from them covariant, coordinate-independent information about gravitational field. Our canonical gravitational superenergy tensor and our canonical averaged relative energy-momentum tensor are exactly the quantities of such a kind. In application to gravitational radiation these quasilocal quantities unambiguously show that any real gravitational wave always transfer the energy and momentum.

Thus, the negative conclusions given by the authors which have used pseudotensors and complexes in an arbitrary coordinates are *incorrect* [1, 2].

Finally, we would like to emphasize that the our local or, at most, quasilocal results are complementary to the very old global results obtained by A. Trautman [8] in an asymptotically flat spacetime which admitted outgoing gravitational radiation.

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